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THE UNIVERSITY OF ALBERTA

TWO MODELS OF NUCLEAR ROTATIONS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

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ABSTRACT

In this thesis we present two simple models of nuclear rotations.

The first model consists of a mass-point rotating in a stationary potential. This potential was deducted from the energy relation of the Harris model. The results of the quantized version of this model are compared to experiments.

The second model is the field theoretical equivalence of particles confined to a circle of a fixed radius. The particles interact through an interaction dependent on their relative angle and their angular momentum.

Approximations of the energy eigenvalues of the first model do fit quite well to the experimental results for the ground state rotational bands of typical rotational nuclei. Due to the approximation method the fit is not so good for the lowest excited states of each nuclei. The predicted values of the moment of inertia and the quadrupole moment are a bit lower than the experimental ones.

The second model gives a very good fit to experimental results both for high and low states of angular momentum.

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CHAPTER 1

INTRODUCTION

The study of rotational spectra of nuclei has given a valuable insight into collective and single particle motion, and their coupling in the nucleus.

Attempts were made as early as 1938 by Teller and Wheeler to detect rotational energy spectra of few nuclei, but as they expected the rotation to be a property of the whole nucleus, thought of as a rigid rotor with the corresponding classical moment of inertia, and due to inadequate equipment this search was without any success.

In 1951 Bohr put forward the theory that rotational excitation modes of the nucleus were the necessary consequence of a strongly deformed equilibrium state (spontaneous rotational symmetry breaking) (Bohr, 51). The same nuclei should therefore show large quadrupole moments.

Shortly after rotational bands were identified, in observation of level sequences proportional to $\ell(\ell+1)$ (ℓ: angular quantum number) (Bohr and Mottelson, 75 p. 2).

Most of the early observed rotational bands were in agreement with the energy expression:

$$E(\ell) = \frac{\hbar^2}{2} \frac{\ell(\ell+1)}{I} \qquad (\ell = 2,4,6\ldots \text{ even parity})$$

(I: moment of inertia)

due to the fact that highly excited states were not

available at this time.

In nearly all cases the moment of inertia was smaller than predicted by assuming a rigid nuclear structure (Bohr and Mottelson, 75 p. 74).

Bohr explained this deviation from the rigid moment of inertia by assuming that only a fraction of the nuclear matter was participating in the rotational motion around a stationary nuclear core.

Soon it was discovered that all nuclei with $150 \leq A \leq 186$, $A \geq 224$ and some few others, had rotational spectra (Mariscott and Scharff-Goldhaber, 69).

Many nuclei were found to violate the $\ell(\ell+1)$ dependence. With improving methods to excite high angular momentum states in nuclei it was found that in nearly all cases the level spacing was denser than predicted by the $\ell(\ell+1)$ rule.

As a consequence of this it was suggested that a term of the form:

$$- B\ell^2(\ell+1)^2$$

should be added to the energy expression to account for rotation vibrational interaction (Scharff-Goldhaber et.al., 76 p. 245); however with this additional term the energy expression could not be used to fit experimental results for neutron rich nuclei, in which the rotational and vibrational level spacing is approaching the same order of magnitude with increasing N, neutron number (Mariscott

and Scharff-Goldhaber, 69 p. 1865).

In spite of nuclei with widely differing N, z, E(ℓ=z) Mallman was able to show that if the energy ratios E_6/E_2, E_8/E_2 for even-even nuclei were plotted against E_4/E_2 he would obtain two universal curves (Scharff-Goldhaber et.al., 76 p. 269). This seems to indicate that the same dynamical features govern nuclei of all magnitudes of deformation.

One of the first models of nuclear rotation was the hydrodynamical one by Bohr and Mottelson (Scharff-Goldhaber et.al., 76 p. 246). It assumed $\ell(\ell+1)$ energy dependence and was therefore inadequate to describe the energy levels for high ℓ for most nuclei. Moreover in this model one has to assume too high values for the deformation parameter β in order to predict values for the moment of inertia divided by the quadrupole moment comparable to experiment (Prakash et.al., 73 p. 2475).

If rotational bands are assumed to be described by

$$E = \frac{\hbar^2 \ell(\ell+1)}{2I}$$

then the decrease in level spacing with increasing ℓ, must be due to increasing moment of inertia I. It has been observed that for most nuclei I is approaching the value for a rigid nucleus with increasing ℓ (Mariscott et.al., 69 p. 1866).

Many models have been constructed to account for this increase in I.

An early model (β stretching model) tried has the energy expression:

$$E_\ell(\beta) = \frac{1}{2} C(\beta_\ell - \beta_o)^2 + \frac{\ell(\ell+1)}{2I(\beta_\ell)} \quad .$$

I is here in units of $\hbar^2$, β is the deformation parameter. To derive the value of β_ℓ, the equilibrium condition $\partial E_\ell / \partial \beta_\ell = 0$ is used (Mariscott et.al., 69 p. 1866).

This model gave a good fit to rotational bands of strongly deformed nuclei, if one assumed $I \sim \beta^2$ (as in the hydrodynamical model) but was inaccurate for others (Mariscott et.al., 69 p. 1866). One disadvantage of this model is that later studies of the nucleus (muonic x-rays and isomer shift in Mössbauer effect) have shown that β does not increase as much with ℓ as this model would indicate (Mariscott et.al., 69 p. 1866).

A very successful one has been the Variable Moment of Inertia model (V.M.I. model). The energy according to this model is given by:

$$E_\ell(I) = \frac{1}{2} C(I - I_o)^2 + \frac{1}{2} \frac{\ell(\ell+1)}{I}$$

with the equilibrium condition:

$$\frac{\partial E(I)}{\partial I} = 0 \quad .$$

Here the moment of inertia I is treated as a general variable, instead of the deformation parameter (Mariscott et. al., 69 p. 1867). This model has justified the Mallman curves, and given a good energy fit closer to the

spherical region than the β stretching model. The success of the V.M.I. model may be related to the fact that nothing is stated about how I depends on, for example, the β stretching of the nucleus nor about the decrease in pairing energy with increasing ℓ.

Another model, the cranking model, has been shown to give the same results as the V.M.I. model, when expanded to the fourth power of the angular velocity ω by perturbation theory (Mariscott et.al., 69 p. 1870).

The cranking model considers a rotating deformed self-consistent potential well. The hamiltonian in the intrinsic nuclear frame is

$$H_o - \omega L_z \quad .$$

H_o is the static hamiltonian of the nucleus.

The energy expression up to ω^4 in the lab.frame is

$$E_{rot} = \frac{1}{2}\omega^2(I_o + 3C\omega^2)$$

$$(\ell(\ell+1))^{\frac{1}{2}} = \omega(I_o + 2C\omega^2)$$

(Harris, 65). Expression for E_{rot} independent of ω can be obtained from these two expressions. It will have the same form as the energy formula of the V.M.I. model.

Harris has shown that the energy of a rotational band can be approximated by a self-consistency approach to any power of ω

$$E_{rot} = \frac{1}{2}\omega^2(I_o + 3C\omega^2 + 5D\omega^4 + 7F\omega^6 + \ldots)$$

$$(\ell(\ell+1))^{\frac{1}{2}} = \omega(I_o + 2C\omega^2 + 3D\omega^4 + 4F\omega^6 + \ldots)$$

(Harris, 65 p. B511). For the even-even rare earth nuclei Harris was able to fit the energy levels within 0.76% rms with just a two parameter fit, while the experimental error is 0.3%.

Gupta has considered an interesting model, in which an axially symmetric prolate deformed nucleus is rotating around an axis perpendicular to its symmetry axis. In this model the centrifugal stretching is counteracted by a harmonic restoring force (Gupta, 69).

This model gives satisfactory fit to experimental values of the rotational energy of even-even nuclei, though not as good as the Harris model, but instead it gives a suggestion for the internal mechanism in a nucleus at the classical level.

By increasing the strength of the restoring force, Gupta is able to reproduce the characteristics of nuclear vibration of two different modes (ellipsoidal and axial symmetric).

This model was later renewed by adding to it a rotationally invariant core to better account for the internal structure of the nucleus and give the possibility that the moment of inertia could be expanded in a series in terms of ω (Trainor and Gupta, 71) basically different from the expression for I in the hydrodynamical model.

This model, named the governor model, gives satisfactory values of the moment of inertia and the quadrupole moment in contrast to the hydrodynamical model (Prakash et.al., 73).

The governor model has been quantized by Prakash et.al. resulting in a closed form expression for the energy (Prakash et.al., 73) that gives a good fit to experimental values of the energy as before.

It has been observed when the moment of inertia of several nuclei is plotted versus the angular velocity, that the resulting smooth curves "bend back" for some high ℓ values, the graph comes triple valued. Neither the V.M.I. model nor the cranking model according to perturbation theory can explain this phenomenon.

To account for the back bending and some other "irregularities" in the moment of inertia a model that deals with the interactions between the nucleons directly has to be constructed (for back bending see, for example, Scharff-Goldhaber et.al., 76 p. 282).

One model that deals with those difficulties is constructed from the second quantized version of the cranking model hamiltonian

$$H = H_o - \omega L_z$$

$$H_o = \sum_{ij} T_{ij} C_i^\dagger C_j + \frac{1}{4} \sum_{ijk\ell} V_{ijk\ell} C_i^\dagger C_j^\dagger C_k C_\ell \quad .$$

Then a canonical transformation is applied to write H_o in terms of weakly interacting quasiparticles.

The inertial forces introduced by the external cranking potential, $-\omega L_z$, become very strong for states of high angular momentum so perturbation methods are not valid.

This model instead deals with them exactly, so shape transitions can happen at high ℓ resulting in back bending. Moreover this model is not limited to axial deformations and is self-consistent, as it relates the energy spectrum directly to the nucleon-nucleon interactions (Scharff-Goldhaber et.al., 76 p. 301-312). In 1976 only very few calculations with this model including realistic forces had been done, but the preliminary results were promising (Scharff-Goldhaber et.al., 76, p. 314).

Several other models and methods have been tried to describe and explain nuclear rotation, but these examples served to show the mainstreams.

In the first part of this thesis we want to take a new approach to the problem of nuclear rotation.

It is known that the energy expression of the Harris model:

$$E = \frac{1}{2}\omega^2(I_o + 3C\omega^2)$$

describes the rotational energy of even-even nuclei with good accuracy, as mentioned before.

Is it possible to find or construct a one particle potential classically and quantum mechanically that will

give the same expression for E? If so, we can think of this potential as the average potential that the valence nucleons are subjected to, as all experiments seem to indicate that just a part of the nuclear material is participating in the rotation.

This is done in the first part (Chapters 2 and 3), and expectation values of the nuclear radius R, moment of inertia I and the quadrupole moment Q are compared to experimental values and other models.

Thieberger (Thieberger, 70) has introduced a classical model, which has the same energy expression as the Harris model:

Two water reservoirs are connected to each other through a narrow pipe. The smaller one is rotating around the heavier stationary one in a uniform gravitational field.

A model with similar structure has been considered by Bassichis and Foldy (Bassichis and Foldy, 64). Particles are moving on a circle, of a constant radius, interacting through a two body force symmetric in the angular variable $\theta_1 - \theta_2$. If in the second quantized version we just allow for the two lowest energy levels, then we have the lowest level of zero angular momentum and a degenerate level of opposite and equal angular momentum.

In the second part (Chapter 4) of this thesis we search for an interaction, with the Thieberger and Harris models in mind, that would when introduced to the second

quantized model give a similar energy relation as the Harris model.

The second part is concluded by inspection of several symmetries conserved by this model.

CHAPTER 2

SEARCH FOR A CLASSICAL POTENTIAL TO EXPLAIN NUCLEAR ROTATION ENERGIES

In this chapter we will deduce a potential that can be used in a simple classical model of nuclear rotations, from expressions that have been used successfully to describe rotational energy levels in many even-even nuclei.

In some models the variable moment of inertia has been accounted for by assuming that the centrifugal force is counteracted by a harmonic restoring force, as for example in the governor model by Trainor and Gupta (Trainor and Gupta, 71). But to our best knowledge nobody has proceeded the same way we will.

It has been shown by Harris (Harris, 65) and others (Mariscott et.al., 69) that the following formulas for the energy E and the moment of inertia I :

$$E = \frac{1}{2}\omega^2(I_o + 3C\omega^2) + E_o \qquad (2.1)$$

$$I = (I_o + 2C\omega^2) \qquad (2.2)$$

I_o: moment of inertia of the ground state

ω : angular velocity

C : higher order inertial parameter

do give an adequate description of the rotational energy of a wide range of nuclei. By combining the two equations (2.1) and (2.2) we get:

$$E = \frac{1}{2}\omega^2(I + C\omega^4) + E_o \quad . \tag{2.3}$$

Now we want to find a model that has the following equation of motion:

$$m\ddot{r} - mr\dot{\phi}^2 + \frac{\partial}{\partial r} V(r) = 0 \tag{2.4}$$

$V(r)$: central potential

ϕ : angular variable

which when in equilibrium (that is when $\dot{r},\ddot{r} = 0$) will lead to the energy expression (2.3).

In equilibrium the equation of motion (2.4) will have the form:

$$mr\dot{\phi}^2 = \frac{\partial V}{\partial r} \tag{2.5}$$

which when solved for $\dot{\phi}^2$ gives:

$$\dot{\phi}^2 = \frac{1}{mr}\frac{\partial}{\partial r} V \quad . \tag{2.6}$$

The expression (2.2) for the moment of inertia will give when divided by the mass m:

$$r^2 = (r_o^2 + \frac{2C}{m}\dot{\phi}^2) \quad . \tag{2.7}$$

If we use expression (2.6) for $\dot{\phi}^2$ in (2.7) we get the following differential equation for the potential

$$r_{eq}^2 = \left[r_o^2 + 2C\frac{1}{m^2 r}\frac{\partial}{\partial r} V(r)\right]_{r=r(\dot{\phi})}$$

and therefore:

$$\frac{\partial}{\partial r} V(r)\Big|_{eq} = \frac{m^2 r}{2C}(r^2 - r_o^2)\Big|_{r=r(\dot{\phi})} \quad . \tag{2.8}$$

This equation has the exact solution:

$$V(r) = \frac{m^2}{8C} (r^4 - 2r_o^2 r^2) \quad . \tag{2.9}$$

The system that has (2.4) as an equation of motion has the following energy expression when in equilibrium:

$$E_{eq} = [\tfrac{1}{2} mr^2\dot{\phi}^2 + V(r)]_{r=r(\dot{\phi})} \quad . \tag{2.10}$$

If we use equation (2.9) for V(r) in this expression for E we get:

$$E_{eq} = [\tfrac{1}{2} mr^2\dot{\phi}^2 + \frac{m^2}{8C} (r^4 - 2r_o^2 r^2)]_{r=r(\dot{\phi})} \quad . \tag{2.11}$$

Now from the equilibrium condition (2.6) we have the following relationship between r^2 and $\dot{\phi}^2$:

$$\dot{\phi}^2 = \frac{1}{mr} \frac{\partial}{\partial r} V(r) = \frac{m}{2C} (r^2 - r_o^2) \tag{2.12}$$

therefore:

$$r^2 = \frac{2C}{m} \dot{\phi}^2 + r_o^2 \tag{2.13}$$

when this expression of r^2 in terms of $\dot{\phi}^2$ is used in (2.11) we get:

$$E_{eq} = \frac{1}{2} mr_o^2\dot{\phi}^2 + \frac{3}{2} C\dot{\phi}^4 - \frac{1}{8C} r_o^4 = \frac{1}{2} I_o\dot{\phi}^2 + \frac{3}{2} C\dot{\phi}^4 + E_o \tag{2.14}$$

which has exactly the form we wanted (2.1).

So the conclusion is that, the potential that classically does give the wanted energy dependence (2.1) on $\dot{\phi}^2$ is:

$$V(r) = \frac{m^2}{8C} (r^4 - 2r_o^2 r^2) \quad . \tag{2.9}$$

CHAPTER 3

QUANTIZED MODEL FOR NUCLEAR ROTATIONS

In this chapter we will quantize a simple model involving the potential (2.9) which was shown, in the last chapter, to be able to give the successful classical energy expression for nuclear rotation (2.1).

Then the energy eigenvalues of the quantized model will be compared to results of experiments on rotational levels of two typical nuclei.

The predictions of this model for the nuclear radius, the moment of inertia and the quadrupole moment will also be compared to experimental values of these quantities.

The chapter will be concluded by a comparison of our model to some others.

The Model

The model we want to quantize will consist of a spinless mass-point subject to the fixed potential

$$V(r) = \alpha(r^4 - 2r_o^2 r^2) \quad . \tag{3.1}$$

An equivalent picture of the model consists of two mass-points moving around a common mass center.

A quantization axis will be chosen to pass through the coordinate-origin along the azimuthal axis of the coordinate system.

The Energy Eigenvalues

The energy eigenvalues $E_{n\ell}$ of this model are given by the one particle Schrödinger equation for the potential (3.1):

$$U_\ell'' + \frac{2\mu}{\hbar^2}\left(E_{n\ell} - \alpha(r^4 - 2r_o^2 r^2) - \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2}\right) U_\ell = 0 \quad (3.2)$$

ℓ: angular quantum number

μ: mass of the mass-point

n: total quantum number

U_ℓ: $R_\ell \cdot r$, where R_ℓ is the radial wave function.

This equation (3.2) cannot be solved in a closed form. The eigenvalues we need are those corresponding to angular excitation but minimal radial excitation, which means $n_r = 0$ (the radial quantum number) and $n = \ell$, as $n = n_r + \ell$. The approximation method chosen for $E_{n\ell}$ has to be consistent with these conditions on the quantum numbers.

Before going further it is interesting to notice that in the fast-rotation limit when $V(r) \to \alpha r^4$, $E_{n\ell} \to \ldots \ell^{4/3}$ which is the natural limit of the V.M.I. model (Moszkowski, 79 p. 2394). Derivation of this limit for $E_{n\ell}$ is in Appendix B, and is further supported by derivations of $E_{n\ell}$ according to the W.K.B. method (which is valid for $n_r \gg \ell$) in Appendix A.

As we are interested in finding $E_{n\ell} - E_o$ (where E_o is the zero point energy i.e. $n = \ell = 0$) for $n_r = 0$, rather than $E_{n\ell}$, we can use the following approximation:

$$E_{n\ell} - E_o \simeq g(r_{o\ell}) = \alpha(r_{o\ell}^4 - 2r_o^2 r_{o\ell}^2) + \frac{\hbar^2 \ell(\ell+1)}{2\mu r_{o\ell}^2} \qquad (3.3)$$

where $r_{o\ell}$ is defined by the following minimum condition on $g(r)$:

$$\frac{d}{dr} g(r)\Big|_{r=r_{o\ell}} = 0 \quad . \qquad (3.4)$$

$g(r)$ is here the effective potential, that is the potential $V(r)$ (3.1) with the centrifugal term added.

The centrifugal term is not modified here by replacing $\ell(\ell+1)$ by $(\ell+\frac{1}{2})^2$, as we want to have:

$$r_o = r_{o\ell}\Big|_{\ell=0} \quad .$$

This approximation method works well for the harmonic oscillator when $(\ell+\frac{1}{2})^2$ dependence is assumed, where it gives:

$E_\ell - E_o \simeq \ldots (\ell+\frac{1}{2})$ instead of the exact result: $E_\ell - E_{o\ell} \simeq \ldots \ell$.

So we can expect the method to give close to correct eigenvalues for high ℓ.

The minimal condition (3.4) is equivalent to finding the real root of:

$$x^3 - r_o^2 x^2 - \frac{\hbar^2 \ell(\ell+1)}{4\alpha\mu} = 0 \quad . \qquad (3.5)$$

where:

$$x = r_{o\ell}^2 \quad .$$

The Moment of Inertia

The moment of inertia I is defined as:

$$I = \int r^2 d\mu \qquad (3.6)$$

where:

$$d\mu = dv\rho(r,\Omega) \qquad (3.7)$$

dv: volume element

dμ: mass element

$\rho(r,\Omega)$: mass density or probability distribution of the nucleus

Ω : angular variables

As the model is axially symmetric we have:

$$\rho_\ell(r,\Omega) = \mu |Y_{\ell o}|^2 \left|\frac{U_\ell}{r^2}\right|^2 \qquad (3.8)$$

$Y_{\ell o}$: spherical harmonics

μ : the total mass

and therefore:

$$I_\ell = \mu \int |U_\ell|^2 r^2 dr \int Y^*_{\ell o} \sin^2\theta \, Y_{\ell o} \, d\Omega \quad .$$

The radial integral that yields the expectation value of r_ℓ^2: $\langle r_\ell^2 \rangle$ will be approximated by $r_{o\ell}^2$. By doing the angular integration for $\ell = 2$ we get:

$$I_2 \approx \mu r_{o\ell}^2 \, 0.48 \qquad (3.9)$$

The Quadrupole Moment

The quadrupole moment of the model, as it is rotating around a symmetry axis, can be defined as

$$Q = \int z\rho(r,\Omega)\, r^2\, (3\cos^2\theta - 1)\, dv \qquad (3.10)$$

z: number of unit charges e assigned to the mass-point

where:

$$z\rho_\ell(r,\Omega) = z|Y_{\ell o}|^2 \left|\frac{U_\ell}{r^2}\right|^2 \qquad (3.11)$$

is the probability distribution of the charge ze in the model. By making the same approximation for $\langle r^2 \rangle$ as before we get for $\ell = 2$ the following quadrupole moment:

$$Q_2 \approx z r_{o\ell}^2 \frac{4}{7} . \qquad (3.12)$$

Comparison with Experiments

Now we want to see if the rotational bands of our model will have the same structure as the rotational bands of two typical nuclei Hf^{170} and U^{238}. The V.M.I. model does give a good fit to experimental results for these two nuclei.

In our model we will assume that all the nuclear mass is located in the mass-point, though experiments do indicate that not all the nuclear mass does participate in the rotation.

By doing this we will get the minimum values for I_2, Q_2 and the nuclear radius $r_{o\ell}$.

Table 1

Computed and Experimental Values of E_ℓ, Q_2, I_2, $r_{o\ell}$ for Hf^{170}

ℓ	E_{exp} keV	E_{comp} keV	$r_{o\ell}$ fm
0	0	0	2.21
2	101	141	3.2
4	321	355	3.3
6	641	665	3.4
8	1042	1051	3.5
10	1504	1500	3.6
12	2015	2000	3.7
14	2565	2546	3.8
16	3150	3130	3.9
18	3764	3751	4.0
20	4417	4403	4.1

$\alpha = 11.6 \frac{KeV}{fm^4}$

$r_o = 2.21$ fm

$\frac{\hbar^2}{2\mu} = 122.4$ keV fm^2

$\frac{2}{\hbar^2} I_2$ comp MeV^{-1}	$\frac{2}{\hbar^2} I_2$ exp MeV^{-1}	Q_2 exp barn	Q_2 comp barn
40	59.8	~5-7	4.2

Sources: E_ℓ, I_ℓ : (Sayer et.al., 75).

Q_2 : (Scharff-Goldhaber et.al., 75 p. 279).

Table 2

Computed and Experimental Values of

E_ℓ, Q_2, I_2, $r_{o\ell}$ for U^{238}

ℓ	E_{exp} keV	E_{comp} keV	$r_{o\ell}$ fm
0	0	0	3.00
2	45	43	4.4
4	148	161	4.4
6	302	334	
8	518	553	
10	775	810	
12	1076	1098	
14	1414	1414	5.0

$\alpha = 2.45 \frac{keV}{fm^4}$

$r_o = 3.00$ fm

$\frac{\hbar^2}{2\mu} = 87.4$ keV fm^2

Zero point energy of 42 keV was subtracted from E_{comp}.

$\frac{2}{\hbar^2} I_{2\ comp}$ MeV^{-1}	$\frac{2}{\hbar^2} I_{2\ exp}$ MeV^{-1}	$Q_{2\ exp}$ barn	$Q_{2\ comp}$ barn
105	133.6	~11	10.3

Sources: E_ℓ, I_ℓ : (Sayer et.al., 75).

Q_2 : (Scharff-Goldhaber et.al., 75 p. 279).

The hamiltonian of our model is invariant with respect to rotation of 180° around the symmetry axis, so the angular quantum number ℓ will occur in sequences of even numbers (Bohr and Mottelson, 75 p. 8).

In the preceding data fit we have used the following constants:

$\mu = 938 \text{ MeV}/c^2$ for one nucleon

$\hbar = 6.58 \times 10^{-22} \text{ MeV}$

$C = 3.00 \times 10^{23} \text{ fm}$

This fit to experimental results is by no means a "best fit", but is rather intended to show that our model can reproduce the general trends of rotational bands of some typical nuclei. A "best fit" method would be complicated and slow, due to the fact that the Schrödinger equation (3.2) (a boundary value problem) does not have a solution in a closed form, and approximation methods for boundary value problems become time consuming when they have to be iterated many times.

As expected the values for I and Q are lower than the experimental ones due to the assumption that all nuclear mass participates in the rotation.

If we assume that the nuclear radius for the ground state ($\ell = 0$) can be calculated according to the empirical formula:

$$r_o = R A^{1/3} \qquad (3.13)$$

R: $\sim$1.2 fm

A: number of nucleons

then we get:

$$r_o(Hf^{170}) \sim 6.7 \text{ fm}$$

$$r_o(U^{238}) \sim 7.4 \text{ fm}$$

which is nearly twice the radii our model gives with the conditions we did assume.

Comparison with Other Models

Prakash et.al. were able to get a very good fit to experimental energy values by a model that has the following Schrödinger equation (Prakash et.al., 73)

$$(-\frac{\hbar^2}{2\mu}\frac{d^2}{d\rho^2} + \frac{\hbar^2\ell(\ell+1)}{2\mu\rho^2} + \frac{\kappa}{2}(\rho - \rho_o)^2 - E)\psi(\rho) = 0 \quad (3.14)$$

where $\mu = M/4$ is the reduced mass of a system of two mass-points.

This model is a quantized version of a model by Gupta (Gupta, 69), where the nucleus is represented by a prolate ellipsoid, rotating around an axis perpendicular to the symmetry axis. The semi-major axis has the length:

$$R_o(1 - \sqrt{\frac{5}{4\pi}}\,\beta) \quad (3.15)$$

where R_o is the average radius and β is a parameter measuring the deformation from a spherical form. If the ellipsoid is divided into two halves along a plane perpendicular

to the symmetry axis, then it is easily shown that the distance ρ between the two mass centers will be:

$$\rho = \frac{3}{4} R_o (1 - \sqrt{\frac{5}{4\pi}}\ \beta) \quad . \qquad (3.16)$$

By a change of a variable the Schrödinger equation (3.14) can be written:

$$(-\frac{\hbar^2}{2B_T} \frac{d^2}{d\beta^2} + \frac{\hbar^2}{2I_T}\ \ell(\ell+1) + \frac{C_T}{2}\ (\beta - \beta_o)^2 - E)\psi = 0 \qquad (3.17)$$

with

$$B_T = \frac{15}{32} \frac{3MR_o^2}{8\pi}$$

$$C_T = \frac{45}{64\pi} \kappa R_o^2$$

$$I_T = \frac{45}{128} \frac{2MR_o^2}{5}\ (1 + \sqrt{\frac{5}{4\pi}}\ \beta)^2 \qquad (3.18)$$

$$\frac{C_T}{B_T} = \frac{\kappa}{\mu} = \omega_o$$

and finally from (3.14) we have:

$$E_{n_r,\ell} = \frac{\hbar^2}{2\mu\rho_I^2}\ \ell(\ell+1) + \frac{\kappa}{2}\ (\rho_I - \rho_o^2) + (n + \frac{1}{2})\hbar(\frac{3\ell(\ell+1)\hbar^2}{\mu^2\rho_\ell^4} + \frac{\kappa}{\mu})^{\frac{1}{2}} \qquad (3.19)$$

with

$$\rho_I = \rho_o + \ell(\ell+1)\ \frac{\hbar^2}{\mu\kappa\rho_\ell^3} \quad .$$

In their fit to experiment $n_r = 0$ is required, and the zero point energy is subtracted from $E_{n_r,\ell}$.

Due to the complicated angular dependence of the nucleus in the Prakash model, $I_o\ (\ell = 0)$ is calculated instead of I_2.

Table 3

Comparison of ours and the Prakash Model for Hf^{170} with Experimental Values

	$\frac{\hbar^2}{2} I_o$ / MeV^{-1}	$\frac{\hbar^2}{2} I_2$ / MeV^{-1}	r_{o2} / fm
Our Model		40	3.2
Model of Prakash	63.4		2.84
Experimental		59.8	6.6

r_{o2} is here half the distance between the mass-points in the Prakash model, which is equivalent to the radius in our model.

The Prakash model gives a better result for I, as in this model the ground state is assumed to be deformed, which is experimentally verified to be true, but a strange condition in a central potential.

The nuclear radius in the model of Prakash R_o is given by (3.16) and is larger than half the distance between the mass-points due to the spheroidal ground state.

By assuming that not all the nuclear mass is rotating, I_2 and r_{o2} in our model will increase in value.

The main difference between our model and the Prakash one is that Prakash and coworkers take a successful

model developed by Gupta (Gaupta, 69) where a harmonic restoring force is pre-assumed and quantize it. In our case we begin with two equivalent models, the fourth order cranking- and the V.M.I. model, that give a good description of the rotational energy spectrum. Then we arrive at a potential that classically does give the same results, and later quantize the model. Our method has therefore strong similarities to so-called "inverse methods" in physics.

In this work the main emphasis has been on the model itself rather than using its results, as we could not find similar approaches to the problem in the physics literature.

CHAPTER 4

SIMPLE FIELD THEORETICAL MODEL OF NUCLEAR ROTATIONS

In this chapter we take a simple second quantized model describing particles rotating on a circle of a fixed radius (Bassichis and Foldy, 64) and compare it with the Harris model (Harris, 65) and an ingenious classical model by Thieberger (Thieberger, 70) that has the same expressions for I and E as the Harris model.

After this comparison we introduce a more general interaction into the model of Bassichis and Foldy so its energy eigenvalues will have the same properties as the rotational bands of typical even-even nuclei.

This chapter is concluded by observation of several symmetries of this new model.

Description of the Models

In the model of Bassichis and Foldy (Bassichis and Foldy, 64) there are N bosons of equal mass and zero spin with a two body interaction present. Its hamiltonian is:

$$H = -\sum_{i=1}^{N} \frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial \theta_i^2} + \frac{1}{2} \sum_{i \neq j} V(\theta_i, \theta_j) \ . \qquad (4.1)$$

The radius is fixed at a value R. If the potential is assumed to depend only on $|\theta_i - \theta_j|$ then H will have the following form in the second quantized version:

$$H = \sum_k \frac{\hbar^2 k^2}{2mR^2} a_k^\dagger a_k + \sum_{k_1 k_2 k_3 k_4} v(k_1 - k_4)\delta_{k_1+k_2, k_3+k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4} \tag{4.2}$$

where $a_k^\dagger$ and a_k are creation and destruction operators of angular momentum. By assuming that all but the two lowest single particle excitations are negligible and that $v(0)=0$ and $v(1) = g$, then H takes the form:

$$H = \hbar\omega(a_1^\dagger a_1 + a_3^\dagger a_3) + g(a_2^\dagger a_2 (a_1^\dagger a_1 + a_3^\dagger a_3) + a_2^2 a_1^\dagger a_3^\dagger + a_2^{\dagger 2} a_1 a_3) \tag{4.3}$$

the state of no angular momentum is labelled by 2, and the states of angular momentum of magnitude $\pm\hbar$ are labelled by 1 and 3.

This system has two important constants of motion (Bassichis and Foldy, 64 p. A936), the total number of particles N:

$$N = \sum_{i=1}^{3} a_i^\dagger a_i \tag{4.4}$$

and the total angular momentum L of the system:

$$L = a_1^\dagger a_1 - a_3^\dagger a_3 \quad . \tag{4.5}$$

If the number of particles in level 3 is given by n, then the state function is:

$$\psi = \sum_{n=0}^{\frac{1}{2}(N-\ell)} C_n |N, \ell, n\rangle \tag{4.6}$$

where the C_n's must satisfy the normalization condition:

$$\sum_{n=0}^{\frac{1}{2}(N-\ell)} |C_n|^2 = 1 \ . \qquad (4.7)$$

Before going any further let us now look at another model.

Thieberger (Thieberger, 70) has invented a classical mechanical model to describe the successful expressions of the Harris model:

$$E = \frac{1}{2}\omega^2(I_o + 3C\omega^2) \qquad (2.1)$$

$$I = (I_o + 2C\omega^2) \ . \qquad (2.2)$$

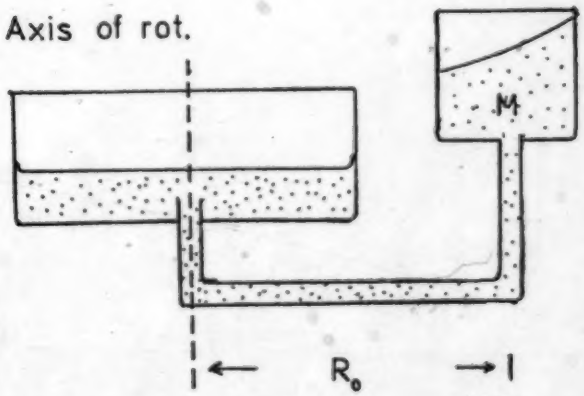

Figure 1: The Thieberger Model

The model consists of two cylindrical reservoirs, the smaller one rotates at a radius R_o with the angular velocity ω, around the stationary bigger one. The liquid in the reservoirs can flow freely through the narrow tube connecting them. The model is stationed in a homogeneous gravitational field. Due to this field and the rotation the potential energy of the system has the form:

$$V = (AM + BM^2/2)\Big|_{M_o}^{M} \tag{4.8}$$

where M_o is the mass in the small reservoir when $\omega = 0$.

Energy Eigenvalues of the Simple Boson Model

It is now possible to find the energy eigenvalues of the hamiltonian (4.3) of the Bassichis and Foldy model. The method for doing this will be used for a more general hamiltonian of our model, and to estimate the order of magnitude of the parameters involved.

For the energy eigenvalues E we have the following expression:

$$\langle\psi|H|\psi\rangle = \langle\psi|E|\psi\rangle \tag{4.9}$$

By using the state function (4.6) expression (4.9) will be equivalent to:

$$\begin{aligned} C_n E = C_n &[\ell + 2n + g(N - \ell - 2n)(\ell + 2n)] \\ &+ gC_{n-1}[(N - \ell - 2n + 2)(N - \ell - 2n + 1)(\ell + n)n]^{\frac{1}{2}} \\ &+ gC_{n+1}[(N - \ell - 2n - 1)(N - \ell - 2n)(\ell + n + 1)(n + 1)]^{\frac{1}{2}} \end{aligned} \tag{4.10}$$

which is an exact difference equation for E, that has to be solved numerically in most cases.

For each ℓ there are several eigenvalues, which correspond to vibrational modes in the nucleus. So for each ℓ we take the lowest eigenvalue, representing a pure

rotational excitation. The lowest eigenvalue can be found by a program involving the Fortran library subroutine EQRT1S, to find the lowest eigenvalue of a tridiagonal matrix. If the corresponding eigenvector is needed the subroutine EQRT2S can be used.

Actually we need the eigenvalues with the zero point energy E ($\ell=0$) subtracted from, to compare with experimental results.

Bassichis and Foldy (Bassichis and Foldy, 64 p.A937) do use a Bogoliubov transformation to estimate the energy of the ground state of the hamiltonian (4.3), we can use this transformation to check the numerical solutions for E (4.10) in the cases of low ℓ's.

In the approximation, a_2 and $a_2^\dagger$ are approximated by $N^{\frac{1}{2}}$, then the hamiltonian (4.3) will have the form:

$$H_B = (a_1^\dagger a_1 + a_3^\dagger a_3)\hbar\omega + gN(a_1^\dagger a_1 + a_3^\dagger a_3 + a_1 a_3 + a_1^\dagger a_3^\dagger) \quad . \qquad (4.11)$$

This hamiltonian (4.11) can be diagonalized by the following canonical transformation:

$$a_1 = C\alpha_1 - S\alpha_3^\dagger \qquad a_3 = C\alpha_3 - S\alpha_1^\dagger$$

$$a_1 = C\alpha_1^\dagger - S\alpha_3 \qquad a_3 = C\alpha_3^\dagger - S\alpha_1 \qquad (4.12)$$

where

$$C = [(\hbar\omega + gN + \varepsilon)/2\varepsilon]^{\frac{1}{2}}$$

$$S = [(\hbar\omega + gN - \varepsilon)/2\varepsilon]^{\frac{1}{2}}$$

$$\varepsilon = [1 + \frac{2gN}{\hbar\omega}]^{\frac{1}{2}}\ \hbar\omega \qquad (4.13)$$

so:

$$C^2 - S^2 = 1 \ .$$

Then the hamiltonian (4.11) takes the form:

$$H_B = \hbar\omega\left(1 + \frac{2gN}{\hbar\omega}\right)^{\frac{1}{2}} - \hbar\omega - gN$$

$$+ \hbar\omega\left(1 + \frac{2gN}{\hbar\omega}\right)^{\frac{1}{2}} (\alpha_1^\dagger\alpha_1 + \alpha_3^\dagger\alpha_3) \quad . \qquad (4.14)$$

So for low ℓ the eigenvalues are approximated by:

$$E_B(\ell) = \hbar\omega\left(1 + \frac{2gN}{\hbar\omega}\right)^{\frac{1}{2}} - \hbar\omega - gN + \hbar\omega\left(1 + \frac{2gN}{\hbar\omega}\right)^{\frac{1}{2}} \ell \qquad (4.15)$$

and therefore:

$$E_B(\ell) - E_B(0) = \ell \cdot \hbar\omega\left(1 + \frac{2gN}{\hbar\omega}\right)^{\frac{1}{2}} \quad . \qquad (4.16)$$

This expression is linear in ℓ, which shows the need for a more general interaction, if the model is going to be used to fit values from experiments for E.

The computed eigenvalues of the hamiltonian (4.3) are shown in comparison with the ones from the Bogoliubov approximation, and experimental values for Hf^{170}, in table 4, just like the Bogoliubov approximation predicted the eigenvalues are almost linear in ℓ.

Table 4

Computed and Approximated Eigenvalues of the Simple Boson Model in Comparison with Experimental Results for Hf^{170}

N = 170 g = 0.8 keV $\hbar\omega$ = 10 keV

ℓ	E_{comp} keV	E_B keV	E_{exp} keV
2	106	106	101
4	209	210	321
6	309	313	642
8	405	415	1041

Comments of Bohr and Mottelson on the Harris Model

Bohr and Mottelson (Bohr and Mottelson, 75 p. 25 Vol. II) note that expansions for the energy E and the moment of inertia I, in terms of the angular velocity ω, occur often naturally in the analysis of rotational properties in terms of the response of nucleonic motion to rotating potentials, and give as a reference (Harris, 65).

For a system rotating around its symmetry axis the expansions can have the following form:

$$I = \alpha + \beta\omega^2 + \gamma\omega^4 + \delta\omega^6 + \ldots. \qquad (4.17)$$

$$E = \frac{1}{2}\alpha\omega^2 + \frac{3}{4}\beta\omega^4 + \frac{5}{6}\gamma\omega^6 + \frac{7}{8}\delta\omega^8 + \ldots. \qquad (4.18)$$

The expansion for $E(\omega)$ has been used successfully to fit results from experiments with high degree of accuracy (Harris, 65).

ω is proportional to the angular momentum ℓ, so the hamiltonian actually can be written in terms of ℓ^2:

$$H_{rot} \sim \ldots \ell^2 + \ldots \ell^4 + \ldots.$$

If the value of ω^2 associated with the quantum state ℓ is obtained from:

$$\hbar^2\ell(\ell+1) = \omega^2 I^2 \qquad (4.19)$$

then:

$$H_{rot} = \frac{\hbar^2}{2\alpha}\ell(\ell+1) - \frac{\hbar^4\beta}{4\alpha^4}(\ell(\ell+1))^2 + \ldots. \qquad (4.20)$$

(Bohr and Mottelson, 75 p. 25).

From this one-body hamiltonian (4.20) and the Thieberger model we can deduce what kind of interaction could be added to the many-body hamiltonian (4.3) for trial.

Model for Nuclear Rotations

As mentioned before Bassichis and Foldy (Bassichis and Foldy, 64) use the hamiltonian:

$$H = -\sum_{i=1}^{N} \frac{\hbar^2}{2mR^2} \frac{\partial^2}{\partial\theta_i^2} + \frac{1}{2} \sum_{i\neq j} V(|\theta_i - \theta_j|) \qquad (4.1)$$

which describes N bosons, on a circle of radius R, with a two body interaction V, to derive the second quantized hamiltonian:

$$H = \hbar\omega(a_1^\dagger a_1 + a_3^\dagger a_3) + g(a_2^\dagger a_2(a_1^\dagger a_1 + a_3^\dagger a_3) + a_2 a_1^\dagger a_3^\dagger + a_2^{\dagger 2} a_1 a_3) \; . \qquad (4.3)$$

The classical hamiltonian for particles on a circle interacting through the two body interaction V is

$$H = \sum_{i=1}^{N} \frac{P_{\theta_i}^2}{2mR^2} + \frac{1}{2} \sum_{i\neq j} V(|\theta_i - \theta_j|) \; . \qquad (4.21)$$

With the Harris and the Thieberger models in mind we want to generalize the interaction in (4.21), by including an interaction that does depend on the angular momentum of the interacting particles.

If this interaction is an even function of the angular momentum then the resulting hamiltonian can be expanded in terms of P_{θ_i} as follows:

$$H = \sum_{i=1}^{N} \frac{P_{\theta_i}^2}{2mR^2} + \frac{1}{2} \sum_{i \neq j} V(|\theta_i - \theta_i|) + \gamma \frac{1}{2} \sum_{ij} \frac{P_{\theta_i} P_{\theta_j}}{2mR^2}$$

$$- \frac{1}{4} \gamma\beta \sum_{ijk\ell} \frac{P_{\theta_i} P_{\theta_j} P_{\theta_k} P_{\theta_\ell}}{4m^2R^4} + \ldots . \qquad (4.22)$$

In the first quantization P_θ is replaced by $-i\hbar \frac{\partial}{\partial\theta}$, this operator is hermitean due to the periodicity of the wave functions. A product of two hermitean operators is also hermitean if they commute. We have:

$$[i\hbar \frac{\partial}{\partial\theta_1}, i\hbar \frac{\partial}{\partial\theta_2}] = 0 \qquad (4.23)$$

so the total H (4.22) is hermitean. (See: Messiah, 59, p. 256).

The second quantization is now performed by introducing the field operators ψ and $\psi^\dagger$, where:

$$\psi_k = \frac{1}{\sqrt{2\pi R}} e^{-ik\theta} a_k \quad . \qquad (4.24)$$

The first two terms of the hamiltonian are derived by Bassichis and Foldy (Bassichis and Foldy, 64) by assuming that $v(k_1 - k_4) = 0$ for all cases except when $k_1 - k_4 = \pm 1$.

For the two body interaction:

$$\gamma \frac{1}{2} \sum_{ij} \frac{P_{\theta_i} P_{\theta_j}}{2mR^2}$$

the matrix element is:

$$-\hbar^2 \frac{1}{2}\gamma \langle \psi_{k_3}\psi_{k_4} | \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2} | \psi_{k_1}\psi_{k_2} \rangle \frac{1}{2mR^2}$$

$$= \frac{1}{2}\frac{\hbar^2}{2mR^2}\gamma \frac{k_1k_2}{(2\pi)^2}\iint d\theta_1 d\theta_2\, e^{-i(k_2-k_3)\theta_1 - i(k_1-k_4)\theta_2}$$

$$\cdot\, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$

$$= \frac{k_1k_2}{4mR^2}\hbar^2\gamma\, \delta_{k_3,k_2}\, \delta_{k_1,k_4}\, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4}$$

$$= \frac{k_1k_2}{4mR^2}\hbar^2\gamma\, a^\dagger_{k_1} a^\dagger_{k_2} a_{k_2} a_{k_1} \; ; \qquad (4.25)$$

for the three level system this will be (as $k_i = \pm k, 0$):

$$\frac{k^2\hbar^2}{2mR^2}\gamma(a_1^\dagger a_1^\dagger a_1 a_1 - 2a_1^\dagger a_1 a_3^\dagger a_3 + a_3^\dagger a_3^\dagger a_3 a_3) \; . \qquad (4.26)$$

The matrix element for the pair interaction is:

$$-\frac{1}{4}\gamma\beta\frac{\hbar^4}{4m^2R^4}\langle k_8k_7k_6k_5 | \frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\theta_3}\frac{\partial}{\partial\theta_4} | k_4k_3k_2k_1\rangle \qquad (4.27)$$

and in much the same way as before this will be (as we get δ_{k_8,k_4}, δ_{k_6,k_3}, δ_{k_6,k_2}, δ_{k_5,k_1}) for the three level system:

$$-\gamma\beta\frac{\hbar^4k^4}{4m^2R^2}(N_1^4 + N_3^4 + 6N_1^2N_3^2 - 4(N_1^3N_3 + N_1N_3^3)) \; . \qquad (4.28)$$

If we set now:

$$gF = \frac{k^2\hbar^2}{2mR^2}\gamma \quad \text{and} \quad g\frac{F}{\alpha} = \gamma\beta\frac{\hbar^4k^4}{4m^2R^4} \qquad (4.29)$$

then the added interaction can be written more simply in terms of the total angular momentum ℓ as:

$$f(\ell) = gF(\ell^2 - \frac{1}{\alpha}\ell^4) \quad . \tag{4.30}$$

The total second quantized hamiltonian of our model will then be:

$$H = \hbar\omega(N_1 + N_3) + g(N_2(N_1 + N_3) + a_2^2 a_1 a_3 + a_2 {}^2 a_1 a_3) + gF(\ell^2 - \frac{1}{\alpha}\ell^4) \tag{4.31}$$

where:

$$\ell = N_1 - N_3.$$

This interaction has to be an even function of ℓ in order to conserve several symmetry properties of the hamiltonian, as will be shown at the end of this chapter.

The interaction $f(\ell)$ does not change the constants of motion, as ℓ is one of them (see: Dirac, 78 p. 92).

In table 5 the energy eigenvalues of H (4.31) are compared to experimental results for Hf^{170}, and in table 6 for U^{238}, with some given choice of the free parameters $\hbar\omega$, g, F, α.

Table 5

Enerty Fit for Hf^{170}

N = 170 F = 18.2

$\hbar\omega$ = 1.5 keV α = 1300

g = 0.8 keV

ℓ	E_{exp} keV	E_{comp} keV
0	0	0
2	101	101
4	321	314
6	642	630
8	1042	1040
10	1504	1528
12	2015	2076
14	2565	2658
16	3150	3249
18	3764	3814
20	4417	4318

The eigenvalues E_{comp} of this model are within 4% of the experimental values, and this fit is by no means a "best fit" of any kind.

Table 6

Energy Fit for U^{238}

N = 238 F = 19.4

$\hbar\omega$ = 0.1 keV α = 1550

g = 0.4 keV

ℓ	E_{exp} keV	E_{comp} keV
0	0	0
2	44.9	44.9
4	148.4	149
6	307.2	310
8	517.7	522
10	775.4	779
12	1076	1073
14	1414	1391

This fit is within 1.6% of the experimental values, and still is not a "best fit".

To get a better fit for Hf^{170} a higher order term in ℓ may be needed for $f(\ell)$.

Moment of Inertia

In this model the radius R, at which the particles are rotating, is a free parameter. It can be fixed for example according to the empirical formula:

$$R = R_o A^{1/3}$$

$$R_o \simeq 1.2 \text{ fm} \quad . \tag{4.32}$$

The state function of the model (4.6) is:

$$\psi = \sum_{n=0}^{\frac{1}{2}(N-\ell)} C_n |N,\ell,n\rangle \tag{4.6}$$

so the moment of inertia I will be:

$$I = mR^2 \sum_{n=0}^{\frac{1}{2}(N-\ell)} |C_n|^2 \cdot n \tag{4.33}$$

where m is the mass of an individual nucleon.

From the normalized eigenvector for the fit of Hf^{170} we get:

$$\sum_{n=0}^{\frac{1}{2}(N-\ell)} |C_n|^2 \cdot n \sim 39 \qquad \text{for} \quad \ell = 2 \tag{4.34}$$

so with R according to (4.32) we have:

$$\frac{2}{\hbar^2} I_{\ell=2} \simeq 83 \text{ MeV}^{-1}$$

while the experimental value is $\sim 60 \text{ MeV}^{-1}$, which we can get by assuming that the radius is 5.7 fm instead of 6.65 fm.

Interpretation of $f(\ell)$

It might be interesting to see what kind of an interaction $f(\ell)$ is in terms of the original operators:

$$N_1 = a_1^\dagger a_1 \qquad N_3 = a_3^\dagger a_3 \ .$$

We know that $\ell = N_1 - N_3$ so:

$$\ell^2 = N_1^2 - 2N_1N_3 + N_3^2$$

$$\ell^4 = N_1^4 - 4N_1^3N_3 + 6N_1^2N_3^2 - 4N_1N_3^3 + N_3^4 \ .$$

The interaction $f(\ell)$ can then be written as:

$$f(\ell) = gF(\ell^2 - \frac{1}{\alpha}\ell^4) = gF(N_1^2 + N_3^2 - \frac{N_1^4}{\alpha} - \frac{N_3^4}{\alpha} - 2N_1N_3$$

$$- \frac{6}{\alpha} N_1^2N_3^2 + \frac{4}{\alpha}(N_1^3N_3 + N_1N_3^3)) \ . \qquad (4.35)$$

We see now that the interaction can be divided into a two body- and a pair interaction both depending on angular momentum. This can be pictured in the following way, where a circle signifies a particle and an arrow the angular momentum vectors.

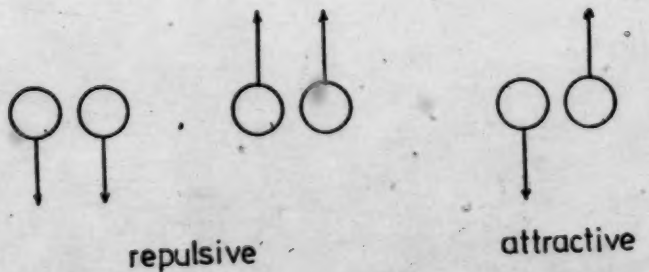

Figure 2: The Two Body Interaction

So the two body interaction has the form of a pairing force to decrease the total angular momentum.

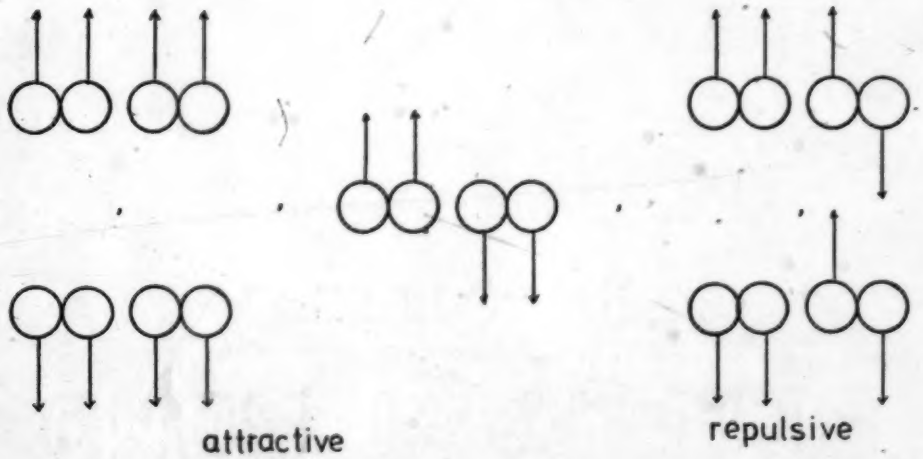

Figure 3: The Pair Interaction

So the weak pair interaction breaks up pairs to increase the angular momentum of the model.

On the level of the original operators the expansion in ℓ has a reassemblance to a multi-polar expansion of a force field.

Symmetry Properties of H

Classically time reversal is defined by:

$$t \to -t \qquad \bar{x} \to \bar{x}$$

$$\Rightarrow \bar{P} \to -\bar{P} \qquad \bar{L} \to -\bar{L}$$

$\bar{P}$ is the linear momentum and $\bar{L}$ is the angular one (Frauenfelder and Henley, 75 p. 418).

In the classical hamiltonian (4.22) L, which is equivalent to P_θ, occurs only in even powers, so these terms are not affected by time reversal. We can therefore conclude that H is invariant under time reversal.

Parity

The conservation of parity yields a quantum number with no classical analogue (Frauenfelder and Henley, 75 p. 359), so we take a look at H (4.28) in the first quantized version.

The parity operator reverses the sign of a polar vector, but leaves an axial one unchanged:

$$\bar{x} \to -\bar{x}$$

$$\bar{p} \to -\bar{p}$$

$$\bar{L} \to \bar{L} \qquad \text{so} \qquad P_\theta \to P_\theta \ .$$

So the hamiltonian H will preserve the parity of the wave functions.

Angular Momentum

We know that the angular momentum ℓ for the second quantized hamiltonian is a constant of the motion, and we expect it is also for the more general classical hamiltonian (4.22).

If L is conserved for H_{cl} then

$$\{H_{cl}, L\} = 0 \qquad \text{where} \qquad \ell = \sum_i P_{\theta_i} \quad . \tag{4.36}$$

With:

$$\{H_{cl}, L\} = \sum_k \left(\frac{\partial H_{cl}}{\partial \theta_k} \frac{\partial L}{\partial P_{\theta_k}} - \frac{\partial H}{\partial P_{\theta_k}} \frac{\partial L}{\partial \theta_k}\right)$$

we have that:

$$\frac{\partial \ell}{\partial \theta_k} = 0 \quad , \quad \frac{\partial \ell}{\partial P_{\theta_k}} = 1 \quad . \tag{4.37}$$

So to prove that the Poisson bracket is equal to zero, we must prove that:

$$\frac{\partial H_{cl}}{\partial \theta_k} = 0 \tag{4.38}$$

which is equivalent to:

$$\sum_k \sum_{i \neq j} \frac{\partial}{\partial \theta_k} V(|\theta_i - \theta_j|) = 0 \quad . \tag{4.39}$$

As the potential V is symmetric in $\gamma_{ij} = \theta_i - \theta_j$ around the origin we can expand $V(|\gamma|)$ in a cosine Fourier series:

$$V(|\gamma_{ij}|) = \sum_n a_n \cos n\gamma_{ij} \quad . \tag{4.40}$$

And:

$$\frac{\partial}{\partial \theta_k} \cos n\gamma_{ki} = - \frac{\partial}{\partial \theta_i} \cos n\gamma_{ki} \quad .$$

So:

$$\sum_k \sum_{i \neq j} \frac{\partial}{\partial \theta_k} V(|\theta_i - \theta_j|) = 0$$

and therefore:

$$\{H_{cl}, L\} = 0 \quad .$$

So just like in the second quantized version the symmetry of the potential ensures the conservation of the angular momentum.

The potential V does not depend on velocity, and H_{cl} does not depend explicitly on t, so the hamiltonian is a constant of the motion and equals the total energy E (Goldstein, 77 p. 220).

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Appendix A

$E_{n\ell}$ according to the W.K.B. Method ($n_r >> \ell$)

In this appendix we want to derive the energy eigenvalues of the Schrödinger equation for the potential $V(r) = \alpha r^4$, by the W.K.B. method (Morse and Feshbach, 53 p. 1101).

The Schrödinger equation is:

$$U_\ell'' + q_r^2 U_\ell = 0 \qquad (A.1)$$

with:

$$q_r^2 = (\varepsilon - v(r) - \frac{\ell(\ell+1)}{r^2}) \qquad (A.2)$$

$$U_\ell = R_\ell r$$

where:

R_ℓ : the radial wavefunction

$\varepsilon = 2\mu E/\hbar^2$

$v(r) = \frac{2\mu}{\hbar^2} V(r)$

μ : the reduced mass of the system .

By making the transformation $r = e^x$ and $U = e^{\frac{1}{2}x}\chi$ the equation (A.1) takes the form:

$$\chi'' + e^{2x}(\varepsilon - v(e^x) - (\ell + \tfrac{1}{2})^2 e^{-2x})\chi = 0 \ . \qquad (A.3)$$

This equation (A.3) can be treated in the same way, by the W.K.B. method, as the one dimensional Schrödinger equation, because the singularities are now at $x = \mp\infty$.

The condition for a bound state according to the W.K.B. approximation is:

$$\int_{x_3}^{x_1} q_x \, dx = (n_r + \tfrac{1}{2})\,\pi \ . \qquad (A.4)$$

If $n_r >> \ell$, x_1, x_3 are positive roots of $q^2 x$ which is the coefficient to χ in eq. (A.3), and n_r is the radial quantum number.

By going back to the variable x we have more simply:

$$\int_{r_3}^{r_1} \left(\varepsilon - v(r) - \frac{(\ell+\frac{1}{2})^2}{r^2}\right)^{\frac{1}{2}} dr = (n_r + \tfrac{1}{2})\pi \ . \qquad (A.5)$$

For $v(r) = ar^4$, q_r^2 will be:

$$q_r^2 = \varepsilon - ar^4 - \frac{(\ell+\frac{1}{2})^2}{r^2} \ . \qquad (A.6)$$

r_1, r_3 will be two positive roots of q_r^2 corresponding to the classical turning points of the effective potential.

$$r^2 q_q^2 = ar^6 - \varepsilon r^2 + (\ell+\tfrac{1}{2})^2 = 0 \ . \qquad (A.7)$$

By changing the variable r, in (A.7), to $x^{\frac{1}{2}}$ we get:

$$x^3 - \frac{\varepsilon}{a} x + \frac{(\ell+\frac{1}{2})^2}{a} = 0 \ . \qquad (A.8)$$

Now there are three possibilities:

1. If $\Delta = \dfrac{(\ell+\frac{1}{2})^4}{4a^2} - \dfrac{\varepsilon^3}{27a^3} > 0$

then there is just one real root.

2. If the same quantity equals zero, then there are three real roots, two of whom are equal.

3. If the same quantity is always larger than zero, then there are three real unequal roots (Selby, 75 p.104).

The 3. possibility can be rewritten as:

$$\varepsilon > (1.61(\ell+\tfrac{1}{2}))^{4/3} a^{1/3}$$

which we will see later is always satisfied.

To see what we can expect from the W.K.B. method, we can easily do the integration for $\ell = 0$:

$$\int_0^{4\sqrt{\varepsilon/a}} (\varepsilon - ar^4)^{\frac{1}{2}} dr = (n_r + \tfrac{3}{4})\pi \quad . \tag{A.9}$$

3/4 gives a better approximation here than 1/2.

Change the variable: $r = \alpha x$, $x = \frac{1}{\alpha} r$, $\alpha = 4\sqrt{\varepsilon/a}$, then we have:

$$(n_r + \tfrac{3}{4})\pi = \int_0^1 (\varepsilon - \varepsilon x^4)^{\frac{1}{2}} dx^4 \sqrt{\varepsilon/a} \tag{A.10}$$

which comes to:

$$\varepsilon = \left(\frac{(n_r + \frac{3}{4})\pi a^{\frac{1}{4}}}{\int_0^1 (1 - x^4)^{\frac{1}{2}} dx} \right)^{4/3} \tag{A.11}$$

$$I = \int_0^1 (1 - x^4)^{\frac{1}{2}} dx = \int_0^1 \frac{1 - x^4}{\sqrt{1 - x^4}} dx \quad .$$

From the integral tables in (Selby, 75) we have:

$$I = \frac{1}{4\sqrt{2\pi}} \Gamma^2 (\tfrac{1}{4}) - \frac{1}{3\sqrt{2}} K (\tfrac{1}{2}) \simeq 0.875$$

where $K(k^2)$ is the complete elliptic integral of the first kind.

Expression (A.11) can now be written in the final form:

$$\varepsilon \simeq \left[\frac{(n_r + \frac{3}{4})\ \pi\ a^{\frac{1}{4}}}{0.875}\right]^{4/3} \quad . \qquad \text{(A.11b)}$$

According to (Selby, 75 p. 104) the roots of q_x^2 can be expressed as

$$\begin{aligned} x_1 &= A + B \\ x_3 &= -\tfrac{1}{2}(A+B) - \tfrac{i\sqrt{3}}{2}(A-B) \\ x_2 &= -\tfrac{1}{2}(A+B) + \tfrac{i\sqrt{3}}{2}(A-B) \quad . \end{aligned} \qquad \text{(A.12)}$$

If:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \left[-\frac{(\ell+\frac{1}{2})^2}{2a} \pm \left(\frac{(\ell+\frac{1}{2})^4}{4a^2} - \frac{\varepsilon^3}{27a^3}\right)^{\frac{1}{2}}\right]^{1/3} \quad . \qquad \text{(A.13)}$$

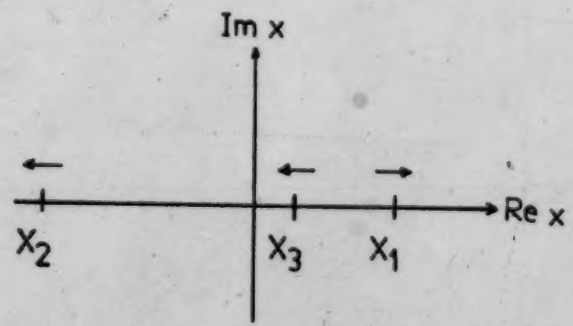

Figure 4: Location of the Roots of q_x^2

The relative location of the roots is shown in figure 4.

The arrows show the trends of the roots as ε increases compared with $(\ell+\frac{1}{2})^2$.

So in order to obtain the energy eigenvalues, we have to evaluate the integral:

$$I = \int (\varepsilon - \alpha r^4 - \frac{(\ell+\frac{1}{2})^2}{r^2})^{\frac{1}{2}} \, dr \quad . \tag{A.14}$$

This can be done in terms of elliptic functions, if we make the following changes: first the integral can be written as:

$$I = \int (\varepsilon r^2 - a r^6 - (\ell+\tfrac{1}{2})^2)^{\frac{1}{2}} \frac{dr}{r} \tag{A.15}$$

then the substitution: $r^2 = x$, or $r = x^{\frac{1}{2}}$, with $dr = \frac{1}{2} x^{-\frac{1}{2}} dx$ will give:

$$I = \frac{1}{2} \int (\varepsilon x - a x^3 - (\ell+\tfrac{1}{2})^2)^{\frac{1}{2}} \frac{dx}{x} \quad . \tag{A.16}$$

By multiplying and dividing the integrand by the square root in it, and taking out a factor of $1/\sqrt{a}$ we get:

$$I = \frac{1}{2\sqrt{a}} \int \frac{\varepsilon x - a x^2 - (\ell+\frac{1}{2})^2}{x\sqrt{\frac{\varepsilon}{a} x - x^3 - \frac{(\ell+\frac{1}{2})^2}{a}}} \, dx \quad . \tag{A.17}$$

The polynomial inside the square root has the root's x_1, x_2 and x_3. In order to write I in the standard form for elliptic integrals we have to write the integrand in terms of the roots x_1, x_2 and x_3.

The limit of integration is between the positive roots x_3 and x_1 so I will have the form:

$$I = \frac{1}{2\sqrt{a}} \int_{x_3}^{x_1} \frac{\varepsilon - a x^2}{\sqrt{(x_1 - x)(x - x_2)(x - x_3)}} \, dx - \frac{1}{2\sqrt{a}} \int_{x_3}^{x_1} \frac{(\ell+\frac{1}{2})^2 \frac{1}{x}}{\sqrt{(x_1 - x)(x - x_3)(x - x_3)}} \, dx$$

$$= I_1 + I_3 \quad . \tag{A.18}$$

To do the integration in I, we use the general formula (Byrd and Friedmann, 71 p. 81)

$$\int_y^a \frac{R(t)\,dt}{\sqrt{(a-t)(t-b)(t-c)}} = g \int_0^{U_1} R(a - (a-b)\,Sn^2U)\,dU \qquad (A.19)$$

which is valid for y in the range $a > y \geq b > c$, R(t) is any polynom in t. Other symbols are defined as follows:

$$g = \frac{2}{\sqrt{a-c}} \qquad SnU_1 = \sqrt{\frac{a-y}{a-b}} \quad .$$

k^2 the parameter of the elliptic functions is defined as:

$$k^2 = \frac{a-b}{a-c} \quad .$$

In our case we have

$$x_1 = a \qquad \text{then} \quad SnU_1 = 1$$

$$x_3 = b = y \qquad \Longrightarrow \quad U_1 = K(k^2) = K$$

$$x_2 = c$$

where K is the complete elliptic integral of the first kind.

I_1 can now be written as:

$$I_1 = \frac{1}{\sqrt{a}\,\sqrt{x_1 - x_2}} \int_0^K (\varepsilon - a(x_1 - (x_1 - x_3)\,Sn^2U)^2)\,dU$$

$$= \frac{1}{\sqrt{a}\sqrt{x_1 - x_2}} \left(\varepsilon K - a \int_0^K (x_1^2 - 2x_1(x_1 - x_3)\,Sn^2U + (x_1 - x_3)^2\,Sn^4U)\,dU\right)$$

$$= \frac{1}{\sqrt{a}\sqrt{x_1 - x_2}} (\varepsilon K - I_2) \quad . \qquad (A.20)$$

From (Byrd and Friedmann, 71 p. 191-192) we have:

$$I_2 = a(Kx_1^2 - 2x_1(x_1-x_3)\frac{1}{k^2}(K - E(K)) + (x_1-x_3)^2\frac{1}{3k^4}((2+k^2)K - 2(1+k^2)E(K)) \quad (A.21)$$

as $E(0,k) = 0$, $Sn0 = 0$, $CnK = 0$. $E(K)$ is the complete elliptic integral of the second kind. Later it will be shown that the coefficient to $E(K)$ is equal to zero.

Now we have to solve the integral:

$$I_3 = -\frac{1}{2\sqrt{a}}\int_{x_3}^{x_1}\frac{(\ell+\frac{1}{2})^2\,dx}{x\sqrt{(x_1-x)(x-x_2)(x-x_3)}} .$$

In our method to solve this integral x_3 must be larger than zero by some small amount at least.

To evaluate I_3 we integrate along the complex contour C shown in figure 5.

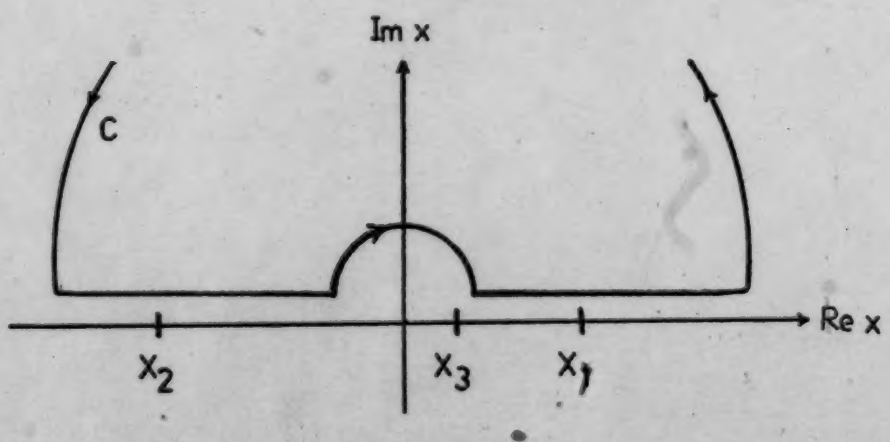

Figure 5: The Contour C

The integration along C, I_C will yield zero as no pole is inside C. The integrand vanishes when the integration in the upper part of C is taken to infinity. So we have:

$$I_C = \int_{-\infty}^{x_2} + \mathcal{P}\int_{x_2}^{x_3} + I_3 + \int_{x_1}^{\infty} - \pi i \operatorname{Res} f\Big|_{x=0} . \qquad (A.22)$$

$\mathcal{P}$ means here principal value and $\operatorname{Res} f\big|_{x=0}$ indicates that the residue value will be taken of the integrand at $x = 0$.

We know that I_3 will be real valued and $I_C = 0$, so as the integrals $\int_{x_2}^{x_3}$ and $\int_{x}^{\infty}$ are purely imaginary, they must cancel each other. Then we have

$$0 = \int_{-\infty}^{x_2} + I_3 - i\pi \operatorname{Res} f\Big|_{x=0}$$

$$= \int_{-\infty}^{x_2} + I_3 + \frac{\pi}{2}(\ell + \tfrac{1}{2}) \quad .$$

So now we can express I_3 as:

$$I_3 = \frac{1}{2\sqrt{a}} \int_{-\infty}^{x_2} \frac{(\ell+\frac{1}{2})^2\, dx}{x\sqrt{(x_1-x)(x-x_2)(x-x_3)}} - \frac{\pi}{2}(\ell+\tfrac{1}{2}). \qquad (A.23)$$

To evaluate this integral we have from (Byrd and Friedmann, 71 p. 70) the general expression

$$I' = \int_{-\infty}^{y} \frac{R(t)\, dt}{\sqrt{(a-t)(b-t)(c-t)}} = g\int_{0}^{U_1} R(a - (a-c)\, nS^2U)\, dU \; . \qquad (A.24)$$

R(t) is any rational function of t. The condition for y is:

$$a > b > c \geq y .$$

And:

$$g = \frac{2}{\sqrt{a-c}} \qquad U_1 = K$$

in our case we identify the roots as follows:

$$x_1 = a$$

$$x_2 = b$$

$$x_3 = c$$

then I_3 can be expressed as:

$$I_3 = \frac{(\ell+\frac{1}{2})^2}{2\sqrt{a}} \frac{2}{\sqrt{x_1-x_3}} \int_0^K \frac{dU}{(x_1 - (x_1-x_2)\, nS^2U)} - \frac{\pi}{2}(\ell+\tfrac{1}{2}) \quad . \quad (A.25)$$

If we now define the parameter $\mu^2 = x_1/(x_1-x_2)$ which is always less by value than one, then the integral itself can be written:

$$-\mu^2 \int_0^K \frac{Sn^2U\, dU}{1-\mu^2 Sn^2U} \qquad \text{with} \qquad nS = \frac{1}{Sn} \qquad (A.26)$$

on the condition that

$$k^2 < \mu^2 < 1 \qquad \text{where} \qquad k^2 = \frac{x_1 - x_3}{x_1 - x_2} .$$

This integral has the solution

$$-\mu^2 \int_0^K \frac{Sn^2U\,dU}{1-\mu^2 Sn^2U} = -\mu^2 \frac{\pi(1 - \Lambda_o(\theta,k))}{2\sqrt{\mu^2(1-\mu^2)(\mu^2-k^2)}} \qquad (A.27)$$

with:

$$\theta = \sin^{-1}\sqrt{\frac{1-\mu^2}{k'^2}} = \sin^{-1}\sqrt{\frac{x_2}{x_2-x_3}} \quad .$$

$\Lambda_o(\theta,k)$ is an elliptic integral of the third kind (Byrd and Friedmann, 71).

After a simple algebraic manipulation I_3 can be written as:

$$I_3 = -\frac{\pi}{2}\frac{(\ell+\frac{1}{2})^2}{\sqrt{a}\sqrt{x_1-x_2}}\frac{(1-\Lambda_o(\theta,k))}{\sqrt{-\frac{x_1x_2x_3}{x_1-x_2}}} - \frac{\pi}{2}(\ell+\tfrac{1}{2}) \quad . \qquad \text{(A.28)}$$

In the polynom q_x^2 the coefficient to x^2 is zero, which imposes the following condition on the roots:

$$x_1 + x_2 + x_3 = 0 \ .$$

In the integral I_2 the coefficient to E(K) is:

$$-\frac{a2x_1}{k^2}(x_1-x_3) + \frac{a(x_1-x_3)^2}{3k^4}2(1+k^2)$$

which can be proven to equal zero, due to the sum condition on the roots, and the fact that $k^2 = x_1-x_3/x_1-x_2$.

The sum condition enables us as well to eliminate x_3 from all our expressions. If now the factor to $K(k^2)$ in integral I_2 is simplified, I will be:

$$I = I_1 + I_3 = \frac{1}{\sqrt{a(x_1-x_2)}}\left[K\{\varepsilon + \frac{a}{3}(x_1^2 + x_1x_2 - x_2^2)\}\right.$$
$$\left. -\frac{\pi}{2}(\ell+\tfrac{1}{2})^2\frac{\{1-\Lambda_o(\theta,k)\}}{\sqrt{\frac{x_1x_2(x_1+x_2)}{x_1-x_2}}}\right] \quad . \qquad \text{(A.29)}$$

And the condition for a bound state in the potential is, as stated before:

$$I - (n_r + \tfrac{1}{2})\pi = 0 \quad .$$

Now we have to check if this expression for bound state energy values, when we require that $\ell = 0$ will yield the same eigenvalues as when $\ell = 0$ was initially assumed for V_{eff}.

If $\ell(\ell+1)$ dependence had been assumed instead of $(\ell+\frac{1}{2})$, then we would have:

$$x_1 \to \sqrt{\varepsilon/a} \qquad x_2 \to -\sqrt{\varepsilon/a}$$

$$x_3 \to 0^+ \qquad \text{as} \quad \ell(\ell+1) \to 0$$

therefore: $k^2 \to 1/2$

and the bound state condition becomes:

$$\sqrt{a}\ \frac{1}{\sqrt{2\sqrt{\varepsilon/a}}} \left[K\left\{\varepsilon + \frac{a}{3}\left(-\frac{\varepsilon}{a}\right)\right\}\right] - (n_r + \tfrac{1}{2})\pi = 0 \qquad \text{(A.30)}$$

which when rearranged will give:

$$\varepsilon \to \left[(n_r + \tfrac{3}{4})\, a^{1/4}\ \frac{3\pi}{\sqrt{2}\ K(\frac{1}{2})}\right]^{4/3}$$

$$\simeq \left[\frac{(n_r + \frac{3}{4})\pi\, a^{\frac{1}{4}}}{0.875}\right]^{4/3} \qquad \text{(A.31)}$$

if $(n_r + \frac{3}{4})$ is assumed instead of $(n_r + \frac{1}{2})$ in bound state condition.

For $n_r >> \ell$ equation (A.29) can be written explicitly for E, as $x_1 \to \sqrt{E/\alpha}$ and the term including Λ_o can be ignored.

$$E = \left((2n_r + \ell + \frac{3}{2}) \frac{\pi\sqrt{2}\cdot 3}{4K(\frac{1}{2})} \right)^{4/3} \left((\frac{\hbar^2}{2\mu})^2 \alpha \right)^{1/3}$$

$$E = \frac{\hbar^2}{2\mu} \varepsilon \quad , \qquad \alpha = \frac{\hbar^2}{2\mu} a \qquad \text{(A.32)}$$

Appendix B

$E_{n\ell}$ in the Limit when $\ell > 1$ and $n_r = 0$ for $V(r) = \alpha r^4$

The energy eigenvalues $E_{n\ell} - E_o$ (the zero point energy E_o has been subtracted here) can be approximated for the potential $V(r) = \alpha r^4$ by:

$$g(r_o) = \alpha r_o^4 + \frac{\hbar^2(\ell + \frac{1}{2})^2}{2\mu r_o^2} \qquad \text{(B.1)}$$

g: is the effective potential: that is, the modified centrifugal term has been added to the potential V(r).

r_o: is the value of the radius at which g(r) takes a minimum value.

if $\ell > 1$ and $n_r = 0$. The minimum value r_o can be shown to be:

$$r_o = \left[\frac{\hbar^2(\ell + \frac{1}{2})^2}{4\alpha\mu}\right]^{1/6} . \qquad \text{(B.2)}$$

If this expression for r_o is now used in (B.1) we get:

$$E\Big|_{n_r=0} = g(r_o) = (\ell+\tfrac{1}{2})^{4/3} \left(\frac{\hbar^2}{2\mu}\right)^{2/3} \alpha^{1/3} \left(\frac{1}{4}^{1/3} + 2^{1/2}\right)$$

$$= (\ell+\tfrac{1}{2})^{4/3}\, 1.890 \left[\left(\frac{\hbar^2}{2\mu}\right)^2 \alpha\right]^{1/3} . \qquad \text{(B.3)}$$